A Borsuk-Ulam type generalization of the Leray-Schauder fixed point theorem and applications

Anatoliy K. Prykarpatsky[(1,2)] and Denis L. Blackmore[(3)]

[(1)] The Dept. of Applied Mathematics at the AGH University of Science and Technology, Kraków 30059, Poland
[(2)] The Ivan Franko State Pedagogical University, Drohobych, Lviv region, Ukraine
[(3)] The Dept. of Mathematical Sciences, New Jersey Institute of Technology, NJ 07102, USA

The classical Leray-Schauder fixed point theorem and its diverse versions [1, 2, 3] in infinite-dimensional both Banach and Frechet spaces have many very important applications in modern applied analysis. Our report is devoted to the operator equation $Ax = f(x)$, where $A : E_1 \to E_2$ is some closed surjective linear operator from Banach space E_1 into Banach space E_2, defined on a domain $D(A) \subset E_1$, and $f : E_1 \to E_2$ is some, in general, nonlinear continuous mapping, whose domain $D(f) \subset D(A) \cap S_r(0)$, with $S_r(0) \subset E_1$ being the sphere of radius $r \in \mathbb{R}_+$ centered at zero. Concerning the mapping $f : E_1 \to E_2$ we will assume that it is $\hat{a}$ -compact. Assume that a mapping $f : E_1 \to E_2$ satisfies the following conditions: 1) the domain $D(f) = D(A) \cap S_r(0)$; 2) the mapping $f : D(f) \to E_2$ is A - compact; 3) there holds a bounded constant $k_f > 0$, such that $\sup\limits_{y \in S_r(0)} \frac{1}{r} \|f(y)\|_2 := k_f$,
where a linear operator $A : E_1 \to E_2$ is taken closed and surjective with the domain $D(A) \subset E_1$. The domain $D(A)$ is not necessary to be dense in E_1. Let now $\tilde{E}_1 := E_1/Ker\ A$ and $p_1 : E_1 \to \tilde{E}_1$ be the corresponding projection. The induced mapping $\tilde{A} : \tilde{E}_1 \to E_2$ with the domain $D(\tilde{A}) := p_1(D(A))$ is defined as usual, that is for any $x \in D(A)$, $p_1(x) \in D(\tilde{A})$ there holds $\tilde{A}\ (p_1(x)) := A\ x$. It is a well known fact that the mapping $\tilde{A} : \tilde{E}_1 \to E_2$ is invertible and its norm is calculated as $k_A^{-1} := \sup\limits_{\|y\|_2=1} \left\| \tilde{A}^{-1}(y) \right\| = \sup\limits_{\|y\|_2=1} \inf\limits_{x \in E_1} \{\|x\|_1 : Ax = y\}$, where we denoted by $\|\cdot\|_1$ and $\|\cdot\|_2$ the corresponding norms in spaces E_1 and E_2. Then the following characteristic theorem holds.

Theorem. *Assume that the dimension* $\dim KerA \geqslant 1$, *and the condition* $k_f < k_A$ *holds; then the quation* $Ax = f(x)$, *possesses on the sphere* $S_r(0) \subset E_1$ *the nonempty solution set* $N(A; f) \subset E_1$, *whose topological dimension* $\dim N(A; f) \geqslant \dim Ker\ A - 1$.

Literatura

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Autor kontaktowy: Anatoliy K. Prykarpatsky
Adres e-mail autora kontaktowego: pryk.anat@ua.fm

Autor referujący: Anatoliy K. Prykarpatsky